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EFFECTIVE MODELS OF COMPOSITE PERIODIC PLATES—III. TWO-DIMENSIONAL APPROACHES

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Abstract—This paper is concerned with two-dimensional approximations of the three-dimensional local problems of Caillerie [(1984). *Math. Meth. Appl. Sci.* **6**, 159–191] and Kohn and Vogelius [(1984). *Int. J. Solids Structures* **20**, 333–350]. The solutions to these problems make it possible to evaluate the effective stiffnesses of periodic plates. The Kirchhoff-type approximation results in the formulae shown by Duvaut [1976. In *Theoretical and Applied Mechanics* (Edited by W. T. Koiter), pp. 119–132. North-Holland, Amsterdam]. By imposing Hencky–Reissner-type constraints, one is led to new formulae which have a wider range of applicability. The paper also discusses the formulae which result from homogeneizing two-dimensional Kirchhoff's, Reissner–Hencky's and Reddy's equations of plates with periodic structure.

1. INTRODUCTION

The asymptotic solution to the statical problem of periodic elastic plates is expressed in terms of the sequence of auxiliary functions which are solutions to the local problems posed by the three-dimensional rescaled cell of periodicity, cf. Parts I and II of this paper. Difficulties that arise in finding these auxiliary functions justify attempts to simplify the problem, e.g. by substituting the appropriate two-dimensional counterparts of the local problems for the original three-dimensional formulations. In the case where the periodicity cell has the shape of a plate, it is expedient to approximate the solutions to the local problems just by means of the methods of modelling developed in the plate theory. Thus, we can apply assumptions of the celebrated plate models of Kirchhoff, Hencky, Reissner and many others. Since the local problems we are faced with are given in their primal formulations, it was thought natural to apply these methods of the theory of plate behaviour modelling which is based upon kinematical constraints (cf. Reissner, 1985; Reddy, 1990; Lewiński, 1987). In Section 2 we prove that impositions of the Kirchhoff-type constraints result in the formulae by Duvaut (1976), originally obtained via homogenizing the equations of bending of periodic plates (cf. Diagram 1). However, it occurs that our passage to the formulae by Duvaut requires a new assumption concerning material properties. In this way do we not only justify these formulae but also determine their range of applicability.

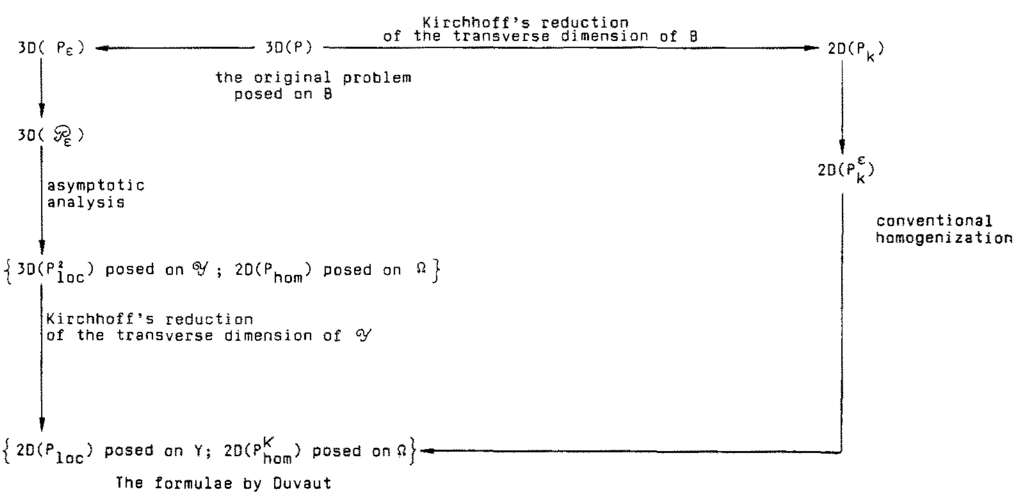

Diagram 1. Under the assumptions formulated in Section 2, this diagram commutes.

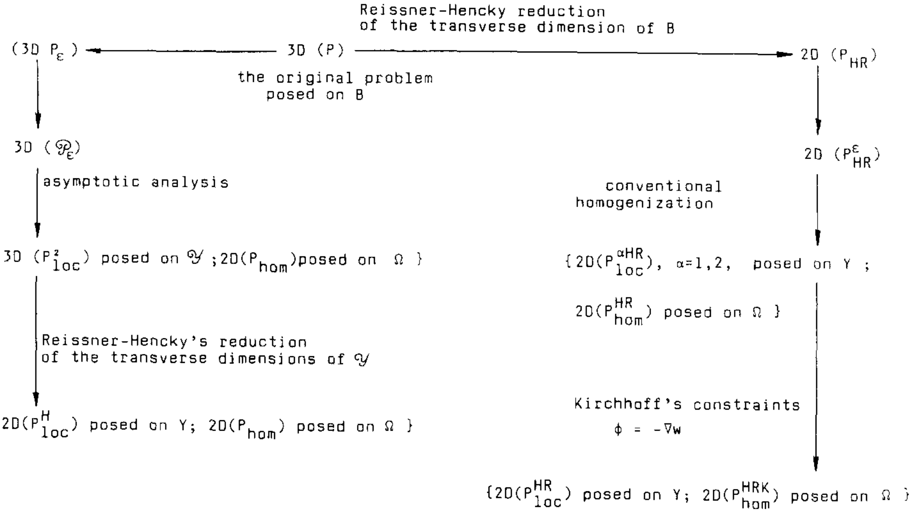

Diagram 2. This diagram does not commute.

The formulae obtained by Duvaut have been found by homogenizing two-dimensional Kirchhoff equations. Similarly, one can homogenize Reissner–Hencky's equations (Section 3) or Reddy's equations (Section 4). On the other hand, the three-dimensional basic cell problems can be simplified by Reissner–Hencky's and even by Reddy's methods of reduction of the transverse dimension. These equations can then be transformed into Kirchhoff-type equations by interrelating the averaged rotations φ_α with the transverse deflection w, according to the contraints $\varphi_\alpha = -w_{,\alpha}$. It is one of the purposes of this paper to report that the above operations do not commute. In particular, the formulae following from the homogenization of Reissner–Hencky's equations do not coincide with those that follow from Reissner–Hencky's approximation of the three-dimensional basic cell problems (cf. Diagram 2).

Because of great complexity of the problem, the evaluation of the approaches discussed needs further detailed studies. A comparison study of Sections 6 and 7, concerning plates with one-directional variation of stiffnesses, is a first stage of this analysis.

The denotations of boundary value problems [e.g. (P^2_{loc})] are common for all parts of this paper. The indices i, j, k, l, m, n takes values 1, 2, 3; the Greek indices α, β, λ, μ, γ, δ, σ, ρ run over 1 and 2; the index ε indicates that the quantity depends upon the small parameter ε; and the indices a, b, c, d assume values 1 and 3. Partial differentiation with respect to x_i and y_i is denoted briefly by $(\ )_{,i}$ and $(\ )_{|i}$, respectively. Summation conventions are used as previously. Averages over Y ands $\mathcal{Y}$ are respectively denoted by $\{\cdot\}$ and $\langle\cdot\rangle$; these parentheses are not used for any other meaning.

2. JUSTIFICATION OF DUVAUT'S FORMULAE FOR EFFECTIVE STIFFNESSES

The subject of consideration is a plate, symmetric with respect to its middle surface $x_3 = 0$ and made from an elastic non-homogeneous material, the properties of which vary Z-periodically with respect to the in-plane variable. The faces of the plate are wavy surfaces that also vary Z-periodically. Such a plate has been described at the beginning of Section 2 of Part II. In this paper, however, we focus our attention only on the bending response of this plate. Moreover, we restrict our analysis to the plates such that

(i) their periodicity cells $\mathcal{Z}$ can be viewed as plates, i.e. their in-plane dimensions are much greater than their maximum thickness;
(ii) the quotients $c_Z^{\alpha\beta} = C_Z^{33\alpha\beta}/C_Z^{3333}$ are constant.

In the following, our consideration will be based upon (P_ε) and $(\mathcal{P}_\varepsilon)$ formulations. Since the cells $\mathcal{Z}$ and $\mathcal{Y}$ are homothetic, assumption (i) implies that $\mathcal{Y}$ also has the shape of a plate. Moreover, condition (ii) implies that

$$c^{\alpha\beta} = C^{33\alpha\beta}/C^{3333}$$

is constant.

The main difficulty in evaluating the effective bending stiffnesses $D_{\text{hom}}^{\alpha\beta\lambda\mu}$ of the plate from the (P_ε) problem, cf. eqn (51) of Part II (to obtain the stiffnesses of the original plate one should take $\varepsilon = \varepsilon_0$) is finding solutions to the three-dimensional basic cell problem (P^2_{loc}), cf. Section of Part I (in the case considered, $\hat{y}_3 = y_3$). Prior to further simplifications, it is expedient to decompose these unknown auxiliary functions as follows:

$$\Xi^{(\alpha,\beta)} = \bar{\Xi}^{(\alpha\beta)} + \tilde{\Xi}^{(\alpha\beta)}. \tag{1}$$

The fields $\bar{\Xi}^{(\alpha\beta)} \in W(\mathscr{Y})$ satisfy

$$(\bar{P}^2_{\text{loc}}) \qquad a(\bar{\Xi}^{(\alpha\beta)}, \mathbf{w}) + \langle y_3 \tilde{C}^{ij\alpha\beta} w_{i|j} \rangle = 0 \tag{2}$$

for every $\mathbf{w} \in W(\mathscr{Y})$, the fields $\tilde{\Xi}^{(\alpha\beta)} \in W(\mathscr{Y})$ being solutions to the variational equation

$$(\tilde{P}^2_{\text{loc}}) \qquad a(\tilde{\Xi}^{(\alpha\beta)}, \mathbf{w}) + \left\langle y_3 \frac{C^{\alpha\beta 33} C^{33ij}}{C^{3333}} w_{i|j} \right\rangle = 0 \tag{3}$$

for every $\mathbf{w} \in W(\mathscr{Y})$.

Upon adding the above equations and considering the definition (60) from Part II of tensor $\tilde{\mathbf{C}}$ and equality (1), one comes back to the original equation (47) in Part I. In virtue of assumption (ii), the $(\tilde{P}^2_{\text{loc}})$ problem possesses the following trivial solution

$$\tilde{\Xi}^{(\alpha\beta)}_k = -\tfrac{1}{2}\delta_{3k}(y_3)^2 c^{\alpha\beta}. \tag{4}$$

Thus the (P^2_{loc}) problem reduces here to solve the $(\bar{P}^2_{\text{loc}})$ problem. Note that in view of (i), the $(\bar{P}^2_{\text{loc}})$ problem can be interpreted as a bending problem for the initially stressed plate $\mathscr{Y}$, with periodical boundary conditions at its lateral surface.

The quantities $y_3\tilde{C}^{ij\alpha\beta}$ represent these initial stresses. By virtue of this interpretation, the solutions to the $(\bar{P}^2_{\text{loc}})$ problem can be viewed as displacement fields in the interior of the plate $\mathscr{Y}$ and furthermore, can be approximated similarly as displacements of a Kirchhoff thin plate. It will be convenient to assume a distribution of displacements according to a Nordgren-like hypothesis (cf. Nordgren, 1971; Lewiński, 1987). Thus we postulate that scalar functions $\chi^{(\alpha\beta)} \in H^2_{\text{per}}(Y)$ exist, such that

$$\bar{\Xi}^{(\alpha\beta)}_\lambda = -y_3\chi^{(\alpha\beta)}_{|\lambda}, \quad \bar{\Xi}^{(\alpha\beta)}_3 = \chi^{(\alpha\beta)} + \tfrac{1}{2}c^{\sigma\mu}(y_3)^2\chi^{(\alpha\beta)}_{|\sigma\mu}. \tag{5}$$

The space $H^2_{\text{per}}(Y)$ consists of the functions from $H^2(Y)$, such that their values and the values of their first derivatives are equal at opposite sides of Y (cf. Duvaut, 1976). When $(\alpha\beta)$ is fixed, the function $\chi^{(\alpha\beta)}$ is the deflection of the middle plane of $\mathscr{Y}$. The test displacements, however, are assumed to obey the Kirchhoff constraints

$$w_\lambda = -y_3 w_{|\lambda}, \quad w_3 = w. \tag{6}$$

On imposing constraints (5) and (6) on the $(\bar{P}^2_{\text{loc}})$ problem and rearranging appropriate terms, one obtains

$$\langle (y_3)^2[\tilde{C}^{\gamma\delta\lambda\mu}\chi^{(\alpha\beta)}_{|\lambda\mu} - \tilde{C}^{\gamma\delta\alpha\beta}]w_{|\gamma\delta}\rangle = 0. \tag{7}$$

Let us define the bending stiffnesses of the rescaled cell as

$$\hat{D}^{\gamma\delta\lambda\mu}(y) = \int_{-c(y)}^{c(y)} (y_3)^2 \cdot \tilde{C}^{\gamma\delta\lambda\mu}(\mathbf{y}) \, dy_3. \tag{8}$$

Integrating with respect to y_3 in eqn (7), one finds

$$\{[\hat{D}^{\gamma\delta\lambda\mu}(y)\chi^{(\alpha\beta)}_{|\lambda\mu} - \hat{D}^{\gamma\delta\alpha\beta}(y)]w_{|\gamma\delta}\} = 0, \tag{9}$$

where $\{\cdot\}$ denotes averaging over Y (cf. Section 4 of Part I). Note that the bending stiffnesses for a plate of thickness $2\varepsilon c$, as considered in the (P_ε) problem, are given by

$$D^{\gamma\delta\alpha\beta}(x/\varepsilon) = \varepsilon^3 \hat{D}^{\gamma\delta\alpha\beta}(x/\varepsilon).$$

The stiffnesses of the original plate [from the (P) problem] are equal to $D^{\gamma\delta\alpha\beta}(x/\varepsilon_0)$ since $\varepsilon_0 Y = Z$.

Thus instead of the (P^2_{loc}) problem, the following two-dimensional problem should be solved:

find $\chi^{(\alpha\beta)} \in H^2_{\text{per}}(Y)$ such that

$$(P^D_{\text{loc}}) \qquad \{[D^{\gamma\delta\lambda\mu}(y)\chi^{(\alpha\beta)}_{|\lambda\mu} - D^{\gamma\delta\alpha\beta}(y)]w_{|\gamma\delta}\} = 0 \tag{10}$$

for every $w \in H^2_{\text{per}}(Y)$.

The above problem coincides with the basic cell problem obtained by Duvaut (1976) by homogenizing Kirchhoff's equations for thin periodic plates. Now insert the approximate solution obtained for the (P^2_{loc}) problem,

$$\begin{aligned} \Xi^{(\alpha\beta)}_\gamma &= -y_3\chi^{(\alpha\beta)}_{|\gamma}, \\ \Xi^{(\alpha\beta)}_\beta &= \chi^{(\alpha\beta)} - \tfrac{1}{2}c^{\sigma\mu}(y_3)^2[\delta^\alpha_\sigma\delta^\beta_\mu - \chi^{(\alpha\beta)}_{|\sigma\mu}], \end{aligned} \tag{11}$$

into the definition of the bending stiffnesses [cf. eqns (51) of Part II, $(55)_4$, (50), (51) of Part I] of the plate from the (P_ε) problem:

$$D^{\gamma\delta\alpha\beta}_{\text{hom}} = t\cdot\varepsilon^3\langle(y_3)^2C^{\gamma\delta\alpha\beta} + y_3C^{\gamma\delta kl}\Xi^{(\alpha\beta)}_{k|l}\rangle, \tag{12}$$

where $t = |\mathscr{Y}|/|Y|$. One is then led to

$$D^{\gamma\delta\alpha\beta}_h = t\cdot\varepsilon^3\langle(y_3)^2[\tilde{C}^{\gamma\delta\alpha\beta} - \tilde{C}^{\gamma\delta\sigma\mu}\chi^{(\alpha\beta)}_{|\sigma\mu}]\rangle, \tag{13}$$

where the $\tilde{C}^{\gamma\delta\alpha\beta}$ are the moduli of the generalized plane stress problem [cf. eqn (60) of Part II]. Upon integrating with respect to y_3, introducing denotations (8) and using the relation $\mathbf{D} = \varepsilon^3\hat{\mathbf{D}}$, we arrive at Duvaut's formula for the effective stiffnesses

$$D^{\gamma\delta\alpha\beta}_h = \{D^{\gamma\delta\alpha\beta}(y) - D^{\gamma\delta\sigma\mu}(y)\chi^{(\alpha\beta)}_{|\sigma\mu}\}; \tag{14}$$

for $\varepsilon = \varepsilon_0$, the above formula defines the stiffnesses of the original Z-periodic plate. Since the variational equation (10) holds true for $w = \chi^{(\sigma\mu)}$, the stiffnesses (14) possess the following symmetries:

$$D^{\gamma\delta\alpha\beta}_h = D^{\delta\gamma\alpha\beta}_h = D^{\delta\gamma\beta\alpha}_h = D^{\alpha\beta\delta\gamma}_h. \tag{15}$$

Moreover, the tensor (14) is positively definite, and hence the homogenized problem is still elliptic (cf. Duvaut, 1976).

Duvaut's effective model for periodic plates can be directly obtained from the variational equation (30) in Part I. In the case of plates with periodically varying thickness, such a derivation has been presented by Kohn and Vogelius (1984). In the slightly more general case considered here, this derivation proceeds as follows. The unknown displacements are assumed to obey the following constraints:

$$U^\varepsilon_\alpha = -\varepsilon y_3 w_{,\alpha} + \varepsilon^2 y_3 \chi^{(\lambda\mu)}_{|\alpha} w_{,\lambda\mu},$$
$$U^\varepsilon_3 = w - \varepsilon^2[\chi^{(\lambda\mu)} + \underline{\tfrac{1}{2}c^{\sigma\rho}(y_3)^2\chi^{(\lambda\mu)}_{|\sigma\rho} - \tfrac{1}{2}c^{\lambda\mu}(y_3)^2}]w_{,\lambda\mu}. \tag{16}$$

In the above formulae, the terms that do not contribute to the effective equations have been omitted. The test functions **V** are assumed to obey Kirchhoff's constraints

$$V_\alpha = -\varepsilon y_3 v_{,\alpha}, \quad V_3 = v, \quad v = v(x). \tag{17}$$

Substitution of formulae (16) and (17) into eqn (30) of Part I results in the variational equation

$$D^{\gamma\delta\lambda\mu}_h \int_\Omega w_{,\lambda\mu} v_{,\gamma\delta}\, \mathrm{d}x = \int_\Omega (q^* v - m^*_\alpha v_{,\alpha})\, \mathrm{d}x, \tag{18}$$

where $D^{\gamma\delta\lambda\mu}_h$ are given by eqns (14), while q^* and m^*_α have been defined by (54) of Part II. The homogenized problem (P^D_{hom}) consists of finding $w \in H^2_0(\Omega)$ such that eqn (18) holds for every $v \in H^2_0(\Omega)$. This problem differs from the $(\bar{P}^A_{\mathrm{hom}})$ problem, elaborated on in Section 2 of Part II, in the definition of the stiffnesses which are now less exactly assessed.

Equation $(16)_1$ does not include the last term of eqn $(3.4)_1$ from Kohn and Vogelius (1984). This term is redundant in the above approach since the virtual fields **V** obey eqns (17). Although the constraints imposed on $\mathbf{U}^\varepsilon$ and **V** are quite different, the bilinear form on the left-hand side of eqn (18) is still symmetric, cf. eqns (15). The derivation of the (P^D_{hom}) problem might have started from assumptions (16) with the underlined terms omitted. However, in that case the assumption of negligibility of stresses σ^{33} should have been additionally stipulated.

3. HOMOGENIZING REISSNER–HENCKY'S EQUATIONS OF PERIODIC PLATES IN BENDING

As in the previous section, we consider the bending problem of a symmetric Z-periodic plate composed of identical segments, $\mathscr{Z}$. The bending problem of such a plate can be approximated by Reissner–Hencky's plate model. This model can then be a departure point for homogenization. This section is aimed at deriving the formulae for the effective stiffnesses of the original Z-periodic plate in this way.

The bending and shearing stiffnesses of the original plate are given by

$$D^{\alpha\beta\lambda\mu}_Z(x) = \int_{-x_3^+(x)}^{x_3^+(x)} (x_3)^2 \tilde{C}^{\alpha\beta\lambda\mu}_Z(x, x_3)\, \mathrm{d}x_3,$$
$$H^{\alpha\beta}_Z(x) = \kappa \int_{-x_3^+(x)}^{x_3^+(x)} C^{\alpha 3\beta 3}_Z(x, x_3)\, \mathrm{d}x_3,$$

since here, $x_3^+ = -x_3^-$. The shear-correction factor κ can be assumed according to the suggestions which can be found in papers concerning moderately thick plates. In the transversely homogeneous case, $\kappa = \frac{5}{6}(1{,}0072)^{-1} \cong 0.827376$ (cf. Ladevèze and Pécastaings, 1988; appropriate formulae for laminated plates can be found in Hinton and Owen, 1984).

According to Reissner–Hencky's approach, the deformations of the original plate subjected to the loads $p_i^{Z\pm}$ are determined by the fields $(w, \boldsymbol{\varphi} = (\varphi_\alpha))$, representing the deflection and rotations of the transverse cross-sections of the plate. In the case of a clamped edge considered here, the problem consists of finding $(w, \boldsymbol{\varphi}) \in V_1(\Omega) = H_0^1(\Omega) \times [H_0^1(\Omega)]^2$ such that

$$(P_{HR}) \int_\Omega [D_Z^{\alpha\beta\lambda\mu}(x)\varphi_{\lambda,\mu}\psi_{\alpha,\beta} + H_Z^{\alpha\beta}(x)(\varphi_\alpha + w_{,\alpha})(\psi_\beta + v_{,\beta})]\,\mathrm{d}x$$

$$= \int_\Omega (m_\alpha^Z \psi_\alpha + q^Z v)\,\mathrm{d}x, \quad \text{for every } (v, \boldsymbol{\psi}) \in V_1(\Omega);$$

where

$$m_\alpha^Z = x_3^+(p_\alpha^{Z+} - p_\alpha^{Z-})(x, x)(G(x/\varepsilon_0))^{1/2},$$
$$q^z = (p_3^{Z+} + p_3^{Z-})(x, x) \cdot (G(x/\varepsilon_0))^{1/2}.$$

The stiffnesses $\mathbf{D}_Z$ and $\mathbf{H}_Z$ and the densities of loads $p_i^{Z\pm}(x, \cdot)$ and $G(\cdot/\varepsilon_0)$ are Z-periodic functions. Using asymptotic homogenization, one can determine an efficient algorithm for computing the displacements and stresses in such a plate. The original plate is considered to be one from the family of εY-periodic plates with stiffnesses

$$\tilde{D}^{\alpha\beta\lambda\mu}\left(\frac{x}{\varepsilon}\right) = \int_{-h(x/\varepsilon)}^{h(x/\varepsilon)} (x_3)^2 \tilde{C}^{\alpha\beta\lambda\mu}\left(\frac{x_3}{\varepsilon_0^{-1}}, \frac{x}{\varepsilon}\right) \mathrm{d}x_3,$$
$$H^{\alpha\beta}\left(\frac{x}{\varepsilon}\right) = \kappa \int_{-h(x/\varepsilon)}^{h(x/\varepsilon)} C^{\alpha3\beta3}\left(\frac{x_3}{\varepsilon_0^{-1}}, \frac{x}{\varepsilon}\right) \mathrm{d}x_3. \quad (19)$$

For $\varepsilon = \varepsilon_0$, the stiffnesses $\mathbf{D}_Z$ and $\tilde{\mathbf{D}}$ and $\mathbf{H}_Z$ and $\mathbf{H}$ coincide. Instead of one problem (P_{HR}), we consider a family of problems which consists of finding $(w^\varepsilon, \boldsymbol{\varphi}^\varepsilon) \in V_1(\Omega)$ such that

$$(P_{HR}^\varepsilon) \int_\Omega \left[\tilde{D}^{\alpha\beta\lambda\mu}\left(\frac{x}{\varepsilon}\right)\varphi_{\lambda,\mu}^\varepsilon \psi_{\alpha,\beta} + H^{\alpha\beta}\left(\frac{x}{\varepsilon}\right)(\varphi_\alpha^\varepsilon + w_{,\alpha}^\varepsilon)(\psi_\beta + v_{,\beta})\right]\mathrm{d}x$$

$$= \int_\Omega (m_\alpha^\varepsilon \psi_\alpha + q^\varepsilon v)\,\mathrm{d}x, \quad \text{for every } (v, \boldsymbol{\psi}) \in V_1(\Omega); \quad (20)$$

where

$$m_\alpha^\varepsilon = h\left(\frac{x}{\varepsilon}\right) \cdot (p_\alpha^+ - p_\alpha^-)\left(x, \frac{x}{\varepsilon}\right)\left[G\left(\frac{x}{\varepsilon}\right)\right]^{1/2},$$
$$q^\varepsilon = (p_3^+ + p_3^-)\left(x, \frac{x}{\varepsilon}\right) \cdot \left[G\left(\frac{x}{\varepsilon}\right)\right]^{1/2}.$$

Both problems, (P_{HR}) and (P_{HR}^ε), are elliptic (cf. Lewiński and Telega, 1988a; Telega and Lewiński, 1988), and hence the homogenization formulae can be obtained by the standard technique developed by Bensoussan *et al.* (1978) for elliptic systems. The basic cell problem here splits into two independent local problems (cf. Lewiński and Telega, 1988a; Tadlaoui and Tapiero, 1988):

find $\mathbf{\Upsilon}^{(\gamma\delta)} \in [H^1_{per}(Y)]^2$ such that

$$(P^{1HR}_{loc}) \quad \{\tilde{D}^{\alpha\beta\sigma\mu}(y)[\Upsilon^{(\gamma\delta)}_{\sigma|\mu} + \delta^\gamma_\sigma \delta^\delta_\mu] v_{\alpha|\beta}\} = 0 \quad \text{for every } \mathbf{v} \in [H^1_{per}(Y)]^2; \tag{21}$$

find $T^{(\delta)} \in H^1_{per}(Y)$ such that

$$(P^{2HR}_{loc}) \quad \{H^{\alpha\beta}(y) \cdot [T^{(\delta)}_{|\beta} + \delta^\delta_\beta] u_{|\alpha}\} = 0 \quad \text{for every } u \in H^1_{per}(Y), \tag{22}$$

where

$$H^1_{per}(Y) = \{v \in H^1(Y) \mid v \text{ assumes equal values at opposite sides of } Y\}.$$

The solutions of the above local problems exist and are defined up to additive constants. Having solved the above problems, one can evaluate the effective stiffnesses

$$D^{\alpha\beta\gamma\delta}_H = \left(\frac{\varepsilon_0}{\varepsilon}\right)^3 \{D^{\alpha\beta\sigma\mu}(y)[\delta^\gamma_\sigma \delta^\delta_\mu + \Upsilon^{(\gamma\delta)}_{\sigma|\mu}]\},$$

$$H^{\alpha\delta}_H = \{H^{\alpha\beta}(y) \cdot [\delta^\delta_\beta + T^{(\delta)}_{|\beta}]\}. \tag{23}$$

The homogenized problem consists of finding the fields $(w^0, \boldsymbol{\varphi}^0) \in V_1(\Omega)$ such that the variational equation of the form (20) holds, where the stiffnesses are constant and given by eqns (23) and the loadings

$$\tilde{m}_\alpha(x) = \{h(y) \cdot (p^+_\alpha - p^-_\alpha)(x, y) \cdot [G(y)]^{1/2}\}$$

$$\tilde{q}(x) = \{(p^+_3 + p^-_3)(x, y) \cdot [G(y)]^{1/2}\}$$

substitute for the loadings m^ε_α and q^ε.

4. HOMOGENIZING REDDY'S EQUATIONS OF BENDING OF PERIODIC PLATES

The bending problem addressed in Section 3 can be dealt with in a similar manner but within the framework of Reddy's (1984) improved version of Reissner–Hencky's plate model. Below we shall not use the original Reddy formulation, but its refined formulation in which the Reissnerian unknowns are involved (Lewiński, 1986). In this version, the boundary value problem of the clamped εY-periodic transversely homogeneous plate consists of finding $(w^\varepsilon, \boldsymbol{\varphi}^\varepsilon) \in H^2_0(\Omega) \times [H^1_0(\Omega)]^2 = V_2(\Omega)$ such that

$$(P^\varepsilon_R) \int_\Omega \left[\tilde{D}^{\alpha\beta\lambda\mu}\left(\frac{x}{\varepsilon}\right) \varphi^\varepsilon_{\lambda,\mu} \psi_{\alpha,\beta} + \frac{1}{84} \tilde{D}^{\alpha\beta\lambda\mu}\left(\frac{x}{\varepsilon}\right) (\varphi^\varepsilon_{\alpha,\beta} + w^\varepsilon_{,\alpha\beta})(\psi_{\lambda,\mu} + v_{,\lambda\mu}) \right.$$

$$\left. + H^{\alpha\sigma}\left(\frac{x}{\varepsilon}\right) (\varphi^\varepsilon_\sigma + w^\varepsilon_{,\sigma})(\psi_\alpha + v_{,\alpha}) \right] \mathrm{d}x = \int_\Omega [m^\varepsilon_\alpha \psi_\alpha + q^\varepsilon v] \, \mathrm{d}x \quad \text{for every } (v, \boldsymbol{\psi}) \in V_2(\Omega). \tag{24}$$

The right-hand side of eqn (24) has been assumed to be in the same form as in the (P^ε_{HR}) problem. This form should be a little more complicated but this difference is of minor interest later on. Since the problem (P^ε_R) is well-posed and elliptic (cf. Bielski and Telega, 1988), the conventional scheme of asymptotic homogenization can be applied. The auxiliary basic cell problem is composed of two independent problems, each being composed of two coupled variational equations.

The functions $(\mathbf{K}^{(\alpha\beta)}, L^{(\alpha\beta)}) \in W_{per}(Y) = [H^1_{per}(Y)]^2 \times H^2_{per}(Y)$ are solutions to

$$(P^{1R}_{\text{loc}})\quad \begin{aligned} &b(\mathbf{K}^{(\alpha\beta)},\mathbf{v})+e(\mathbf{v},L^{(\alpha\beta)})+\{\tilde{D}^{\alpha\beta\lambda\mu}v_{\lambda,\mu}\} = 0\\ &e(\mathbf{K}^{(\alpha\beta)},v)+d(L^{(\alpha\beta)},v) = 0 \quad \text{for very } (\mathbf{v},v)\in W_{\text{per}}(Y). \end{aligned} \tag{25}$$

The functions $(\mathbf{M}^{(\alpha\beta)}, N^{(\alpha\beta)} \in W_{\text{per}}(Y)$ are solutions to

$$(P^{2R}_{\text{loc}})\quad \begin{aligned} &b(\mathbf{M}^{(\alpha\beta)},\mathbf{v})+e(\mathbf{v},N^{(\alpha\beta)})+\tfrac{1}{84}\{\tilde{D}^{\alpha\beta\lambda\mu}v_{\lambda,\mu}\} = 0\\ &e(\mathbf{M}^{(\alpha\beta)},\mathbf{v})+d(N^{(\alpha\beta)},v)+\tfrac{1}{84}\{\tilde{D}^{\alpha\beta\lambda\mu}v_{,\lambda\mu}\} = 0 \quad \text{for every } (\mathbf{v},v)\in W_{\text{per}}(Y), \end{aligned} \tag{26}$$

where the bilinear forms are defined by

$$\begin{aligned} b(\mathbf{v},\mathbf{w}) &= \tfrac{85}{84}\{\tilde{D}^{\alpha\beta\lambda\mu}(y)v_{\alpha,\beta}w_{\lambda,\mu}\},\\ e(\mathbf{v},w) &= \tfrac{1}{84}\{\tilde{D}^{\alpha\beta\lambda\mu}(y)w_{,\lambda\mu}v_{\alpha,\beta}\},\\ d(u,v) &= \tfrac{1}{84}\{\tilde{D}^{\alpha\beta\lambda\mu}(y)u_{,\lambda\mu}v_{,\alpha\beta}\}. \end{aligned} \tag{27}$$

Well-posedness of the above local problems is assured by arguments similar to that used in the paper by Lewiński and Telega (1988b) concerning the homogenization of the thin shell equations. Having found the solutions of the problems $(P^{\alpha R}_{\text{loc}})$, $\alpha = 1, 2$, one can determine the following auxiliary tensor fields:

$$\begin{aligned} A_1^{\alpha\beta\lambda\mu} &= \tilde{D}^{\alpha\beta\delta\gamma}(y)[\delta^\lambda_\delta\,\delta^\mu_\gamma + K^{(\lambda\mu)}_{\delta|\gamma}]\\ A_2^{\alpha\beta\lambda\mu} &= \tilde{D}^{\alpha\beta\delta\gamma}(y)M^{(\lambda\mu)}_{\delta|\gamma}\\ A_3^{\alpha\beta\lambda\mu} &= \tfrac{1}{84}\tilde{D}^{\alpha\beta\delta\gamma}(y)[K^{(\lambda\mu)}_{\delta|\gamma}+L^{(\lambda\mu)}_{|\delta\gamma}]\\ A_4^{\alpha\beta\lambda\mu} &= \tfrac{1}{84}\tilde{D}^{\alpha\beta\delta\gamma}(y)[\delta^\lambda_\delta\,\delta^\mu_\gamma + M^{(\lambda\mu)}_{\delta|\gamma}+N^{(\lambda\mu)}_{|\delta\gamma}] \end{aligned} \tag{28}$$

and with their help, one can find the stiffness tensors

$$(\mathbf{D}_{Rh}, \mathbf{E}_{Rh}, \mathbf{F}_{Rh}, \mathbf{G}_{Rh}, \mathbf{H}_h) = \{\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \mathbf{A}_4, \mathbf{H}\}. \tag{29}$$

The homogenized constitutive relationships are of the form

$$\begin{aligned} M_h^{\alpha\beta} &= D_{Rh}^{\alpha\beta\lambda\mu}\,\varphi^0_{\lambda,\mu} + E_{Rh}^{\alpha\beta\lambda\mu}\,\mu^0_{\lambda\mu},\\ S_h^{\alpha\beta} &= F_{Rh}^{\alpha\beta\lambda\mu}\,\varphi^0_{\lambda,\mu} + G_{Rh}^{\alpha\beta\lambda\mu}\,\mu^0_{\lambda\mu},\\ Q_h^{\alpha} &= H_h^{\alpha\gamma}(w^0_{,\gamma}+\varphi^0_\gamma), \end{aligned} \tag{30}$$

where $\mu^0_{\lambda\mu} = \varphi^0_{\lambda,\mu}+w^0_{,\lambda\mu}$; $\boldsymbol{\varphi}^0$ and w^0 being the first terms of the asymptotic expansions of the fields $\boldsymbol{\varphi}^\varepsilon$, w^ε. The homogenized problem consists of finding $(w^0, \boldsymbol{\varphi}^0) \in V_2(\Omega)$ such that

$$\int_\Omega [M_h^{\alpha\beta}\psi_{\alpha,\beta} + S_h^{\alpha\beta}(\psi_{\alpha,\beta}+v_{,\alpha\beta}) + Q_h^\alpha(v_{,\alpha}+\psi_\alpha)]\,\mathrm{d}x = \int_\Omega [\tilde{m}_\alpha\psi_\alpha + \tilde{q}v]\,\mathrm{d}x$$

$$\text{for every } (v, \boldsymbol{\psi}) \in V_2(\Omega). \tag{31}$$

It is worth emphasizing that the homogenized relations $(30)_{1,2}$ are coupled, while the initial relations are uncoupled, as follows from the form of the left hand-side of eqn (24).

5. A NEW HOMOGENIZED MODEL FOR PLATES WITH PERIODIC STRUCTURE

The approximations that have led us to formulae (10) and (14) by Duvaut (1976) can be applied only when the periodicity cell has the shape of a thin plate. Only then can

Kirchhoff's assumptions model the plate behaviour correctly. Thus, a natural way of relaxing the previous approximations is to substitute Hencky's constraints for the previously applied Kirchhoff assumptions. Such an approach could be justified if the cell $\mathscr{Y}$ has the shape of a moderately thick plate. It is expedient to impose the following constraints upon the unknown local fields $\bar{\Xi}^{(\alpha\beta)}$, cf. the $(\bar{P}^2_{\rm loc})$ problem,

$$\begin{aligned}\bar{\Xi}^{(\alpha\beta)}_\lambda &= y_3\Psi^{(\alpha\beta)}_\lambda + \tfrac{1}{6}c^{\sigma\mu}(y_3)^3\Psi^{(\alpha\beta)}_{\mu|\sigma\lambda},\\ \bar{\Xi}^{(\alpha\beta)}_3 &= W^{(\alpha\beta)} - \tfrac{1}{2}c^{\sigma\mu}(y_3)^2\Psi^{(\alpha\beta)}_{\mu|\sigma},\end{aligned} \tag{32}$$

where $\boldsymbol{\Psi}^{(\alpha\beta)} = (\Psi^{(\alpha\beta)}_\lambda = \Psi^{(\beta\alpha)}_\lambda)$ and $W^{(\alpha\beta)} = W^{(\beta\alpha)}$ are Y-periodic functions. We assumed additionally that $c^{\delta\mu}$ is constant. Thus, the solution of the $(P^2_{\rm loc})$ problem is assumed to be of the form (1) along with (4) and (32). We impose the Hencky-type constraints

$$w_\alpha = y_3 \cdot \psi_\alpha(y), \quad w_3 = v(y) \tag{33}$$

on the test functions $\mathbf{w}$, where $\psi_\alpha,\, v \in H^1_{\rm per}(Y)$.

Substituting (32) and (20) into eqn (2), we find

$$\langle (y_3)^2[\tilde{C}^{\lambda\mu\gamma\delta}\Psi^{(\alpha\beta)}_{\gamma|\delta} + \tilde{C}^{\alpha\beta\lambda\mu}]\psi_{\lambda|\mu} + C^{\sigma 3\lambda 3}(\Psi^{(\alpha\beta)}_\lambda + W^{(\alpha\beta)}_{|\lambda})(\psi_\sigma + v_{|\sigma})\rangle + \underline{O(\varepsilon^5)} = 0 \tag{34}$$

for every $v, \psi_\alpha \in H^1_{\rm per}(Y)$. The underlined term of higher order will be discarded. Note that the above equation involves only first derivatives of the functions $\Psi^{(\alpha\beta)}_\lambda$ and $W^{(\alpha\beta)}$, hence they may be sought in the space $H^1_{\rm per}(Y)$ [we can forget about formulae (32) to which the second derivatives of $\Psi^{(\alpha\beta)}_\lambda$ contribute]. Upon integration with respect to y_3, one is led to the local problem:

find $(\boldsymbol{\Psi}^{(\alpha\beta)}, W^{(\alpha\beta)}) \in [H^1_{\rm per}(Y)]^2 \times H^1_{\rm per}(Y)$ such that

$$(P^H_{\rm loc}) \quad \{[\hat{D}^{\lambda\mu\gamma\delta}\Psi^{(\alpha\beta)}_{\gamma|\delta} + \hat{D}^{\alpha\beta\lambda\mu}]\psi_{\lambda|\mu} + \hat{H}^{\sigma\lambda}(\Psi^{(\alpha\beta)}_\lambda + W^{(\alpha\beta)}_{|\lambda})(\psi_\sigma + v_{|\sigma})\} = 0 \quad \text{for every } \psi_\alpha, v \in H^1_{\rm per}(Y), \tag{35}$$

where $\hat{\mathbf{H}} = \mathbf{H}/\varepsilon_0$; $\mathbf{H}$ being defined by $(19)_2$ where a shear correction factor has been introduced.

The above problem possesses a mathematical form similar to the form of the (P_{HR}) problem (Section 3) and with the help of this fact, one can corroborate that the solution to $(P^H_{\rm loc})$ exists and is determined up to additive constants (which can depend upon x). Note the essential difference between the local problems discussed in Section 3 and herein. The problems $(P^{1HR}_{\rm loc})$ and $(P^{2HR}_{\rm loc})$ are decoupled while eqn (35) cannot be split into two independent variational equations.

Having found the solution to the $(P^H_{\rm loc})$ problem, we determine the fields (32), add them to (4) and insert them into definition (12) of $\mathbf{D}_{\rm hom}$. Thus, we find a new approximate formula for the effective bending stiffnesses

$$D^{\alpha\beta\gamma\delta}_{zh} = \{D^{\alpha\beta\sigma\mu}(y)[\delta^\gamma_\sigma\delta^\delta_\mu + \Psi^{(\gamma\delta)}_{\sigma|\mu}]\}, \tag{36}$$

where $\mathbf{D} = \varepsilon^3\hat{\mathbf{D}}$. The similarity of the above definitions and (23) is worth emphasizing. Note that the fields $W^{(\alpha\beta)}$ do not enter (36) explicitly. Yet note that formula (36) is not symmetric. Just upon symmetrization of (36), the $W^{(\alpha\beta)}$ fields will enter the definition of $\mathbf{D}_{zh}$. To find this, let us substitute $\boldsymbol{\psi} = \boldsymbol{\Psi}^{(\sigma\rho)}$ and $v = W^{(\sigma\rho)}$ into eqn (35), multiply both sides of this identity by ε^3, make use of the relations $\mathbf{D} = \varepsilon^3\hat{\mathbf{D}}$, $\mathbf{H} = \varepsilon_0\hat{\mathbf{H}}$ and add to definition (36) to obtain a definition for $\mathbf{D}_{zh}$ in a new form:

$$D_{zh}^{\alpha\beta\sigma\rho} = \{D^{\lambda\mu\gamma\delta}[\Psi_{\delta|\gamma}^{(\alpha\beta)} + \delta_\delta^\alpha \delta_\gamma^\beta][\Psi_{\lambda|\mu}^{(\sigma\rho)} + \delta_\lambda^\sigma \delta_\mu^\rho]\} + \frac{\varepsilon^3}{\varepsilon_0}\{H^{\mu\lambda}(\Psi_\lambda^{(\alpha\beta)} + W_{|\lambda}^{(\alpha\beta)})(\Psi_\mu^{(\sigma\rho)} + W_{|\mu}^{(\sigma\rho)})\}, \tag{37}$$

from which the symmetries of the type (15) are readily seen. Moreover, formula (37) makes it possible to quite easily prove that the tensor $\mathbf{D}_{zh}$ is positively definite, which is the key to the proof of ellipticity of the homogenized problem. The proof is similar to the proof of positive definiteness of the tensor $\mathbf{A}_z$, cf. Section 5, Step 2, Part I.

6. EFFECTIVE BENDING STIFFNESSES OF PLATES PERIODIC IN ONE DIRECTION

Consider the plate described in Section 2 of Part II, such that $h(y) = h(y_1) = h^+(y_1) = -h^-(y_1), y_1 = x_1/\varepsilon$. Let $Y_1 = a$ and $\varepsilon_0 = \varepsilon$ to simplify notation. The moduli do not vary in the y_2 direction, $C^{ijkl} = C^{ijkl}(y_3, y_1)$; $C^{ijkl}(y_3, \cdot)$ are a-periodic functions. Moreover, $C^{ijkl}(y_3, y_1) = C^{ijkl}(-y_3, y_1)$. As in Section 2 of Part II, we restrict our considerations to the case in which $x_3 =$ const are planes of material symmetry, i.e. $C^{\alpha 3\beta\gamma} = C^{333\gamma} = 0$. Moreover, we assume that x_1 and x_2 are orthotropy axes of the material, viz. $C^{1112} = C^{2221} = 0$.

In the case considered, the two-dimensional local problems reduce to one-dimensional ones and hence may be solved analytically, and the effective bending stiffnesses can be found in closed forms.

6.1. *Formulae due to Duvaut based upon homogenization of Kirchhoff's equations*

Effective bending stiffnesses of the plate considered have been found by Duvaut (1976). They can be written as

$$\begin{aligned} D_h^{1111} &= \{(D^{1111})^{-1}\}^{-1}, \quad D_h^{1212} = \{D^{1212}\} \\ D_h^{1122} &= \{D^{1122}/D^{1111}\}D_h^{1111}, \quad D_h^{1112} = D_h^{1222} = 0 \\ D_h^{2222} &= \{D^{2222} - (D^{2211})^2/D^{1111}\} + \{D^{1122}/D^{1111}\}^2 \cdot D_h^{1111}, \end{aligned} \tag{38}$$

where

$$\{f\} = \frac{1}{a}\int_0^a f(y_1)\,\mathrm{d}y_1.$$

The stiffnesses $D^{\alpha\beta\lambda\mu}(x_1/\varepsilon)$ are given by $(19)_1$. They are εa-periodic functions in x_1, with the function $D^{\alpha\beta\lambda\mu}(\cdot)$ being a-periodic.

6.2. *Formulae based on homogenizing Reissner–Hencky's equations*

The local problems $(P_{\mathrm{loc}}^{\alpha HR})$, $\alpha = 1, 2$, can be solved analytically. Omitting the derivation, we report here only the final results:

$$\begin{aligned} D_H^{1111} &= D_h^{1111}, \quad D_H^{1212} = \{(D^{1212})^{-1}\}^{-1} \\ D_H^{1122} &= D_h^{1122}, \quad D_H^{1112} = D_H^{1222} = 0, \quad D_H^{2222} = D_h^{2222}. \end{aligned} \tag{39}$$

The shearing stiffnesses are

$$H_H^{11} = \{(H^{11})^{-1}\}^{-1}, \quad H_H^{22} = \{H^{22}\}, \quad H_H^{21} = 0. \tag{40}$$

Note that formulae (38) and (39) differ considerably in the definitions of the effective torsional stiffness.

6.3. Formulae based on homogenizing Reddy's equations

The assumption of periodicity in one direction considerably simplifies the $(P^{\alpha R}_{\text{loc}})$ problems, $\alpha = 1, 2$, so that their solutions can be explicitly given. For the sake of brevity, only the final results for effective stiffnesses are reported:

$$
\begin{aligned}
D^{\alpha\beta\lambda\mu}_{Rh} &= D^{\alpha\beta\lambda\mu}_{H} \quad \text{given by eqns (39) (not in general),} \\
E^{1212}_{Rh} &= \tfrac{1}{85}[\{(D^{1212})^{-1}\}^{-1} - \{D^{1212}\}] = F^{1212}_{Rh}, \\
G^{\alpha\alpha\beta\beta}_{Rh} &= \tfrac{1}{84} D^{\alpha\alpha\beta\beta}_{Rh} \quad \text{(do not sum over } \alpha \text{ or } \beta), \\
G^{1212}_{Rh} &= \tfrac{1}{84}[\{D^{1212}\} + E^{1212}_{Rh}], \\
H^{\alpha\beta}_{h} &= \{H^{\alpha\beta}\}.
\end{aligned}
\tag{41}
$$

The other components are equal to the components given above, according to the rules of symmetry, or they vanish.

6.4. Formulae based on Caillerie–Kohn–Vogelius' approach

The key problem is to find solutions of the (P^2_{loc}) problem, cf. Section 5, eqn (47), Part I. Due to restrictions introduced in this section, the vector functions $\Xi^{(\alpha\beta)}$ do not depend upon y_2 here but do possess a special form:

$$\Xi^{(\alpha\alpha)} = (\Xi^{(\alpha\alpha)}_1, 0, \Xi^{(\alpha\alpha)}_3), \quad \Xi^{(12)} = (0, \Xi^{(12)}_2, 0). \tag{42}$$

Thus, the (P^2_{loc}) problem splits into two independent problems posed on the plane domain, still denoted by $\mathscr{Y}$ and parametrized by (y_a), $a = 1, 3$, coordinates. Let

$$V_{\text{per}}(\mathscr{Y}) = \{v \in H^1(\mathscr{Y}) \,|\, v \text{ assumes equal values at opposite sides of } \mathscr{Y}\}.$$

The functions $\Xi^{(\alpha\alpha)}_a \in V_{\text{per}}(\mathscr{Y})$ satisfy

$$(P^{2(\alpha\alpha)}_{\text{loc}}) \qquad \langle [C^{abcd}(y_1, y_3)\Xi^{(\alpha\alpha)}_{c|d} + y_3 C^{ab\alpha\alpha}(y_1, y_3)] w_{a|b} \rangle = 0 \tag{43}$$

for every $w_a \in V_{\text{per}}(\mathscr{Y})$ (do not sum over α) where $a, b, c, d = 1, 3$.

The function $\Xi^{(12)}_2 \in V_{\text{per}}(\mathscr{Y})$ satisfies

$$(P^{2(12)}_{\text{loc}}) \qquad \langle [C^{2a2c}(y_1, y_3)\Xi^{(12)}_{2|c} + y_3 C^{1212}(y_1, y_3)\delta^a_1] w_{|a} \rangle = 0 \tag{44}$$

for every $w \in V_{\text{per}}(\mathscr{Y})$, $a, c = 1, 3$; where $\langle \cdot \rangle$ implies averaging over the two-dimensional domain $\mathscr{Y}$. Having found the functions $\Xi^{(\alpha\alpha)}_a$, $\Xi^{(12)}_2$ (which can be determined up to additive constants), one can evaluate the effective stiffnesses with the help of the formulae

$$
\begin{aligned}
D^{1111}_z &= \langle y_3(C^{1111}\Xi^{(11)}_{1|1} + C^{1133}\Xi^{(11)}_{3|3} + y_3 C^{1111}) \rangle \\
D^{1122}_z &= \langle y_3(C^{1111}\Xi^{(22)}_{1|1} + C^{1133}\Xi^{(22)}_{3|3} + y_3 C^{1122}) \rangle \\
D^{2222}_z &= \langle y_3(C^{2211}\Xi^{(22)}_{1|1} + C^{2233}\Xi^{(22)}_{3|3} + y_3 C^{2222}) \rangle \\
D^{1212}_z &= \langle y_3(C^{1212}(\Xi^{(12)}_{2|1} + y_3)) \rangle.
\end{aligned}
\tag{45}
$$

Other stiffnesses can be found according to symmetry relations of type (15). The stiffnesses are zero when the sum of the indices is an odd number. In general, the $(P^{(2\alpha\alpha)}_{\text{loc}})$ and $(P^{2(12)}_{\text{loc}})$ problems are analytically intractable, and hence only numerical solutions would be available. When the properties of the material differ considerably and discontinuously, the correct numerical solution could then be difficult to find. If one uses finite element codes, special attention should be focused on the appropriate choice of mesh; if available, adaptive techniques can be helpful (cf. Bendsøe and Kikuchi, 1988). Simplifications similar to (42) have been disclosed previously by Kohn and Vogelius (1984) for plates with discontinuously varying thickness.

6.5. *Formulae according to Section* 5

Our first aim is to find the functions $\Psi^{(\alpha\beta)}$ and $W^{(\alpha\beta)}$ which solve the (P^H_{loc}) problem. We conjecture that these functions will be independent of y_2. The differential equations following from eqn (35) assume the form

$$-M^{11(\alpha\beta)}_{|1}+Q^{1(\alpha\beta)}=0,\quad Q^{1(\alpha\beta)}_{|1}=0,\quad -M^{12(\alpha\beta)}_{|1}+\hat{H}^{22}(y_1)\Psi^{(\alpha\beta)}_2(y_1)=0, \tag{46}$$

where

$$\begin{aligned} M^{11(\alpha\beta)} &= \hat{D}^{1111}\Psi^{(\alpha\beta)}_{1|1}+\hat{D}^{11\alpha\beta},\\ M^{12(\alpha\beta)} &= \hat{D}^{1212}\Psi^{(\alpha\beta)}_{2|1}+\hat{D}^{12\alpha\beta},\\ Q^{1(\alpha\beta)} &= \hat{H}^{11}(W^{(\alpha\beta)}_{|1}+\Psi^{(\alpha\beta)}_1). \end{aligned} \tag{47}$$

The above formulae are valid provided that the functions defining the stiffnesses are differentiable. The periodicity conditions which follow from eqn (35) assume the form

$$\Psi^{(\alpha\beta)}_\sigma(0)=\Psi^{(\alpha\beta)}_\sigma(a),\quad W^{(\alpha\beta)}(0)=W^{(\alpha\beta)}(a), \tag{48}$$

$$M^{11(\alpha\beta)}(0)=M^{11(\alpha\beta)}(a),\quad Q^{1(\alpha\beta)}(0)=Q^{1(\alpha\beta)}(a),\quad M^{12(\alpha\beta)}(0)=M^{12(\alpha\beta)}(a). \tag{49}$$

By integrating the first two equations (46) and using the periodicity conditions $(49)_{1,2}$, one finds

$$\Psi^{(\alpha\beta)}_{1|1}=\frac{1}{\hat{D}^{1111}}\left[\{(\hat{D}^{1111})^{-1}\}^{-1}\left\{\frac{\hat{D}^{\alpha\beta11}}{\hat{D}^{1111}}\right\}-\hat{D}^{\alpha\beta11}\right]. \tag{50}$$

The above result enables us to find the effective stiffnesses $D^{\alpha\alpha\beta\beta}_{zh}$. It occurs that

$$D^{\alpha\alpha\beta\beta}_{zh}=D^{\alpha\alpha\beta\beta}_{h}\quad \text{(do not sum over } \alpha \text{ or } \beta\text{)}, \tag{51}$$

where $D^{\alpha\alpha\beta\beta}_h$ have been defined by eqns (38). Moreover, $D^{1222}_{zh}=D^{2111}_{zh}=0$. Thus, only the stiffness D^{1212}_{zh} has not yet been found. Since

$$D^{1212}_{zh}=\{D^{1212}(1+\Psi^{(12)}_{2|1})\}, \tag{52}$$

the problem reduces to finding the function $\Psi^{(12)}_2\in H^1_{\text{per}}(0,a)$ which fulfils the variational equation

$$\{(\hat{D}^{1212}\Psi^{(12)}_{2|1}+\hat{D}^{1212})u_{|1}+\hat{H}^{22}\Psi^{(12)}_2u\}=0\quad \text{for every } u\in H^1_{\text{per}}(0,a); \tag{53}$$

or, in the case where the stiffnesses are of the $C^1(0,a)$ class, the differential equation $(46)_3$ augmented with periodicity conditions (49). In the general case, the solution cannot be given explicitly.

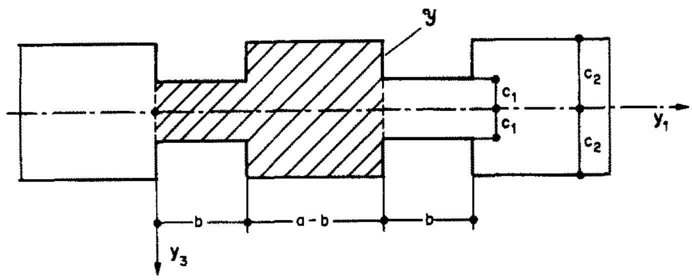

Fig. 1.

7. ANALYSIS OF EFFECTIVE BENDING STIFFNESSES OF PLATES OF RAPIDLY AND PERIODICALLY VARYING THICKNESS

The method outlined in Part I of this paper constructs a path from the original, highly complicated problem (P) to a sequence of simpler problems, mainly to (P_{hom}); viz. to Kirchhoff's problem for a homogenized thin plate. This method (developed by Caillerie, 1984) is based upon very weak assumptions and hence can be treated as being exact, or as a method of reference. Other methods, for instance those described in Sections 2 and 5 which also result in Kirchhoff plate models, provide approximate values of the effective stiffnesses. The method in Section 3 results in a model of a moderately thick plate, but we can impose the constraints $\varphi^0 = -\nabla w^0$ on the equations of this model, leading us again to Kirchhoff's model. Similarly, Reddy's equations can be transformed into the thin plate formulation. Thus for the same composite plate one can obtain various approximate values of the effective stiffnesses.

For the sake of simplicity, we shall analyze below the distribution of these values in the special case of isotropic homogeneous plates of rapidly varying thickness (see Fig. 1). The comparison analysis below can be treated as an extension of the analysis carried out by Kohn and Vogelius (1984). The stiffnesses will be found using the methods set up in Table 1. Results (D) can be obtained via the algorithm described in Section 6.4 or by solving the (P_{loc}^{KV}) problem, cf. Section 5 of Part II, as done by Kohn and Vogelius (1984).

Table 1.

Symbol	Model	Denotation of bending stiffnesses	Range of applicability
A	The Duvaut approach: homogenizing Kirchhoff's equations	$D_h^{\alpha\beta\lambda\mu}$	$\lvert c^+ - c^- \rvert \ll \min Y_\alpha$
B	Reissner-Hencky's plate after homogenization—Section 3	$D_H^{\alpha\beta\lambda\mu}$	$\frac{1}{10} \min Y_\alpha < \lvert c^+ - c^- \rvert < \frac{1}{5} \min Y_\alpha$
C	Reddy's plate after homogenization—Section 4	$D_{Rh}^{\alpha\beta\lambda\mu} = D_H^{\alpha\beta\lambda\mu}$	as in (B)
D	The model of Caillerie–Kohn–Vogelius, cf. Section 5 from Part I and Section 4 from Part II	$D_{\text{hom}}^{\alpha\beta\lambda\mu}$	arbitrary shapes of $\mathscr{Y}$ dim $(\mathscr{Y}) \ll$ dim (Ω)
E	The model proposed in Section 5	$D_{zh}^{\alpha\beta\lambda\mu}$	as in (B)
F	The model based on homogenizing the boundary, cf. Kohn and Vogelius (1984)	$D_{hs}^{\alpha\beta\lambda\mu}$	$\lvert c^+ - c^- \rvert \gg \max Y_\alpha$
G	The effective model based on formulae due to Kączkowski, cf. Section 7	$D_{Kh}^{\alpha\beta\lambda\mu}$	as in (A)
H_A H_D	The effective models based on Huber's formulae eqns (1) from Part I	$D_{\text{Hom}}^{\alpha\beta\lambda\mu}$	reinforced plates or plates stiffened by ribs

Table 2.

No. of table	Dimensions b/a	$\lvert\mathscr{Y}\rvert/a^2$	c_1/a	c_2/a	c_2/c_1	Slenderness λ
3a	1/2	1	1/3	2/3	2	1
3b	3/4	1	1/3	1	3	1
3c	1/2	1	1/4	3/4	3	1
3d	3/4	1	1/4	5/4	5	1
4a	1/2	1/3	2/15	3/15	1.5	3
4b	1/2	1/3	1/9	2/9	2	3
4c	1/2	1/3	1/12	3/12	3	3
4d	3/4	1/3	8/54	12/54	1.5	3
4e	3/4	1/3	4/30	8/30	2	3
4f	3/4	1/3	4/36	12/36	3	3
5a	1/2	1/8	2/40	3/40	1.5	8
5b	1/2	1/8	1/24	2/24	2	8
5c	1/2	1/8	1/32	3/32	3	8
5d	3/4	1/8	2/36	3/36	1.5	8
5e	3/4	1/8	1/20	2/20	2	8
5f	3/4	1/8	4/96	12/96	3	8

Table 3a.

	Model A	B(C)	D	E	F	G	H_A	H_D
$\bar{D}^{1111}$	0.047	0.047	0.031	0.047	0.027	0.047	0.047	0.047
$\bar{D}^{1122}$	0.012	0.012	0.008	0.012	0.007	0.012	0.018	0.0148
D^{2222}	0.114	0.114	0.113	0.114	0.113	0.1184	0.114	0.113
$\bar{D}^{1212}$	0.044	0.0176	0.017	0.0210	0.010	0.0176	0.0137	0.0111

Table 3b.

	Model A	B(C)	D	E	F	G	H_A	H_D
$\bar{D}^{1111}$	0.035	0.035	0.028	0.035	0.026	0.035	0.035	0.028
$\bar{D}^{1122}$	0.009	0.009	0.007	0.009	0.007	0.009	0.0202	0.018
D^{2222}	0.187	0.187	0.187	0.187	0.187	0.197	0.187	0.187
$\bar{D}^{1212}$	0.074	0.013	0.012	0.0144	0.010	0.0130	0.015	0.0136

Table 3c.

	Model A	B(C)	D	E	F	G	H_A	H_D
$\bar{D}^{1111}$	0.0214	0.0214	0.014	0.0214	0.011	0.0214	0.0214	0.014
$\bar{D}^{1122}$	0.005	0.005	0.004	0.005	0.003	0.005	0.014	0.0113
D^{2222}	0.147	0.147	0.147	0.147	0.147	0.155	0.147	0.147
$\bar{D}^{1212}$	0.0583	0.00804	0.011	0.0132	0.004	0.008	0.0105	0.00851

Table 3d.

	Model A	B(C)	D	E	F	G	H_A	H_D
$\bar{D}^{1111}$	0.015	0.015	0.012	0.015	0.011	0.015	0.015	0.012
$\bar{D}^{1122}$	0.004	0.004	0.003	0.004	0.003	0.004	0.0177	0.0158
D^{2222}	0.334	0.334	0.334	0.334	0.334	0.355	0.334	0.334
$\bar{D}^{1212}$	0.133	0.00554	0.006	0.00706	0.004	0.00554	0.0133	0.01187

Table 4.

a	A	B(C)	E	G	H_A
$\bar{D}^{1111}$	2.6 E-3	2.6 E-3	2.6 E-3	2.6 E-3	2.6 E-3
$\bar{D}^{1122}$	6.5 E-4	6.5 E-4	6.5 E-4	6.5 E-4	7.67 E-4
$\bar{D}^{2222}$	3.62 E-3	3.62 E-3	3.62 E-3	3.69 E-3	3.62 E-3
$\bar{D}^{1212}$	1.383 E-4	9.75 E-4	1.204 E-3	9.75 E-4	5.752 E-4
b					
$\bar{D}^{1111}$	1.734 E-3	1.734 E-3	1.734 E-3	1.734 E-3	1.734 E-3
$\bar{D}^{1122}$	4.335 E-4	4.335 E-4	4.335 E-4	4.335 E-4	6.765 E-4
$\bar{D}^{2222}$	4.223 E-3	4.223 E-3	4.223 E-3	4.389 E-3	4.223 E-3
$\bar{D}^{1212}$	1.646 E-3	6.503 E-4	1.166 E-3	6.503 E-4	5.074 E-4
c					
$\bar{D}^{1111}$	7.936 E-4	7.936 E-4	7.936 E-4	7.936 E-4	7.936 E-4
$\bar{D}^{1122}$	1.984 E-4	1.984 E-4	1.984 E-4	1.984 E-4	5.346 E-4
$\bar{D}^{2222}$	5.452 E-3	5.452 E-3	5.452 E-3	5.763 E-3	5.452 E-4
$\bar{D}^{1212}$	2.161 E-3	2.976 E-4	1.141 E-3	2.976 E-4	4.01 E-4
d					
$\bar{D}^{1111}$	2.805 E-3	2.805 E-3	2.805 E-3	2.805 E-3	2.805 E-3
$\bar{D}^{1122}$	7.013 E-4	7.013 E-4	7.013 E-4	7.013 E-4	7.98 E-4
$\bar{D}^{2222}$	3.630 E-3	3.630 E-3	3.630 E-3	3.685 E-3	3.630 E-3
$\bar{D}^{1212}$	1.382 E-3	1.052 E-4	1.162 E-3	1.052 E-3	5.983 E-4
e					
$\bar{D}^{1111}$	2.158 E-3	2.158 E-3	2.158 E-3	2.158 E-3	2.158 E-3
$\bar{D}^{1122}$	5.394 E-4	5.394 E-4	5.395 E-4	5.394 E-4	7.773 E-4
$\bar{D}^{2222}$	4.480 E-3	4.480 E-3	4.480 E-3	4.635 E-3	4.480 E-3
$\bar{D}^{1212}$	1.738 E-3	8.091 E-4	1.028 E-3	8.091 E-4	5.829 E-4
f					
$\bar{D}^{1111}$	1.285 E-3	1.285 E-3	1.285 E-3	1.285 E-3	1.285 E-3
$\bar{D}^{1122}$	3.212 E-4	3.212 E-4	3.212 E-4	3.212 E-4	7.465 E-4
$\bar{D}^{2222}$	6.938 E-3	6.938 E-3	6.938 E-3	7.315 E-3	6.938 E-3
$\bar{D}^{1212}$	2.743 E-3	4.818 E-4	8.038 E-4	4.818 E-4	5.598 E-4

Apart from the approaches discussed previously, we shall also refer to the results reported by Kohn and Vogelius (1984), obtained by homogenizing the boundary (symbol F in Table 1). Moreover, Huber's formulae, eqn (1) in Part I, will be used to assess the stiffnesses D^{1122} and D^{1212} in terms of $D^{\alpha\alpha\alpha\alpha}$, the latter being evaluated according to formulae (38) by Duvaut (1976) (the same values are predicted by homogenizing Reissner–Hencky's equations), in which case the approach is called (H_A), or evaluated according to formulae (45), in which case the approach is called (H_D). We shall also refer to the formulae due to Kączkowski (1980) (symbol G in Table 1), according to which the stiffnesses D_{Kh}^{1111}, D_{Kh}^{2211} are evaluated by eqns $(38)_{1,3}$, D_{Kh}^{1212} according to $(39)_2$ and $D_{Kh}^{2222} = \{D^{2222}\}$.

The non-dimensional values of the effective stiffnesses $\bar{D}^{\alpha\beta\lambda\mu} = D^{\alpha\beta\lambda\mu}/Ea^3\varepsilon^3$ of plates with various shape of periodicity cells (cf. Table 2) are set up in Tables 3a–5f; E is Young's modulus and Poisson's ratio is $\nu = 0.25$. The mean thickness $\bar{h}$ and slenderness of the cell $\mathscr{Y}$ are defined by

$$\bar{h} = \frac{2}{a}[b \cdot c_1 + (b-a) \cdot c_2], \quad \lambda = \frac{a}{\bar{h}} = a^2/|\mathscr{Y}|; \tag{54}$$

where $|\mathscr{Y}| = a \cdot \bar{h}$ is the volume of $\mathscr{Y}$ (per unit depth).

Tables 3a–d present the effective stiffnesses of the plates, the cells of which are such that $\lambda = 1$. The dimensions of the cell $\mathscr{Y}$ lie within the range of applicability of the (D) method. The other results are reported for comparison.

Table 5.

a	A	B(C)	E	G	H_A
$\bar{D}^{1111}$	1.371 E-4	1.371 E-4	1.371 E-4	1.371 E-4	1.371 E-4
$\bar{D}^{1122}$	3.423 E-5	3.423 E-5	3.423 E-5	3.423 E-5	4.044 E-5
$\bar{D}^{2222}$	1.909 E-4	1.909 E-4	1.909 E-4	1.945 E-4	1.909 E-4
$\bar{D}^{1212}$	7.292 E-5	5.143 E-5	6.926 E-5	5.143 E-5	3.033 E-5
b					
$\bar{D}^{1111}$	9.144 E-5	9.144 E-5	9.144 E-5	9.144 E-5	9.144 E-5
$\bar{D}^{1122}$	2.286 E-5	2.286 E-5	2.286 E-5	2.286 E-5	3.570 E-5
$\bar{D}^{2222}$	2.230 E-4	2.230 E-4	2.230 E-4	2.315 E-4	2.230 E-4
$\bar{D}^{1212}$	8.681 E-5	3.429 E-5	7.684 E-5	3.429 E-5	2.677 E-5
c					
$\bar{D}^{1111}$	4.184 E-5	4.184 E-5	4.184 E-5	4.184 E-5	4.184 E-5
$\bar{D}^{1122}$	1.046 E-5	1.046 E-5	1.046 E-5	1.046 E-5	2.818 E-5
$\bar{D}^{2222}$	2.874 E-4	2.874 E-4	2.874 E-4	3.037 E-4	2.874 E-4
$\bar{D}^{1212}$	1.139 E-4	1.569 E-5	9.219 E-5	1.569 E-4	2.114 E-5
d					
$\bar{D}^{1111}$	1.479 E-4	1.479 E-4	1.479 E-4	1.479 E-4	1.479 E-4
$\bar{D}^{1122}$	3.698 E-5	3.698 E-5	3.698 E-5	3.698 E-5	4.210 E-5
$\bar{D}^{2222}$	1.914 E-4	1.914 E-4	1.914 E-4	1.943 E-4	1.914 E-4
$\bar{D}^{1212}$	7.287 E-5	5.549 E-5	6.736 E-5	5.549 E-5	3.155 E-5
e					
$\bar{D}^{1111}$	1.138 E-4	1.138 E-4	1.138 E-4	1.138 E-4	1.138 E-4
$\bar{D}^{1122}$	2.845 E-5	2.845 E-5	2.845 E-5	2.845 E-5	4.10 E-5
$\bar{D}^{2222}$	2.363 E-4	2.363 E-4	2.363 E-4	2.445 E-4	2.363 E-4
$\bar{D}^{1212}$	9.167 E-5	4.267 E-5	7.163 E-5	4.267 E-5	3.075 E-5
f					
$\bar{D}^{1111}$	6.775 E-5	6.775 E-5	6.775 E-5	6.775 E-5	6.775 E-5
$\bar{D}^{1122}$	1.694 E-5	1.694 E-5	1.694 E-5	1.694 E-5	3.936 E-5
$\bar{D}^{2222}$	3.659 E-4	3.659 E-4	3.659 E-4	3.859 E-4	3.659 E-4
$\bar{D}^{1212}$	1.447 E-4	2.541 E-5	8.101 E-5	2.541 E-5	2.952 E-5

Table 4 displays the values of effective stiffnesses [according to methods (A), (B, C), (E), (G) and (H_A)] of plates whose cells have $\lambda = 3$. The cell $\mathscr{Y}$ is not sufficiently slender for the application of methods (A), (B), (E) and (G) to be fully justified.

Table 5 shows the values of the effective stiffnesses such that $\lambda = 8$ according to methods (A), (B, C), (E), (G), (H_A). In this case, application of methods (B, C), (E), (G) and (H_A) is justified.

The greatest discrepancies can be observed in evaluating the torsional stiffnesses D^{1212}. The plots in Figs 2, 3 can help us to compare the values of D^{1212} according to methods (A), (B, C) and (E).

The results obtained make it possible to formulate the following conclusions.

(a) In the case where $\lambda = 1$, the effective stiffnesses should be computed by model (D). This case is not applicable to models (A), (B, C) and (E). Nonetheless, the latter models approximate the stiffnesses $D^{\alpha\alpha\beta\beta}$ quite well—they estimate the relevant values obtained by model (D) from above. However, model (B, C) offers surprisingly good evaluation of the torsional stiffness D^{1212}, while model (A) considerably overestimates this stiffness. Model (F) underestimates the stiffnesses D^{1111}, D^{1122} and D^{1212} in comparison with model (D).

(b) The results for $\lambda > 1$ due to model (D) are unfortunately unavailable but as a reference, the results according model (E) have been taken. As λ increases, the results D_h^{1212} tend to D_{zh}^{1212} from above. This corroborates that Duvaut's formulae are applicable only

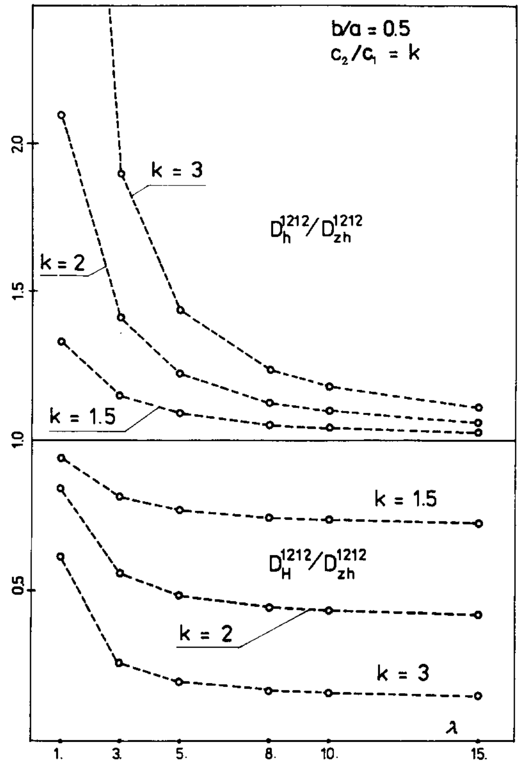
b/a = 0.5
c_2/c_1 = k
k = 3
D_h^{1212}/D_{zh}^{1212}
k = 2
k = 1.5
k = 1.5
D_H^{1212}/D_{zh}^{1212}
k = 2
k = 3
λ
2.0
1.5
1.0
0.5
1.
3.
5.
8.
10.
15.

Fig. 2.

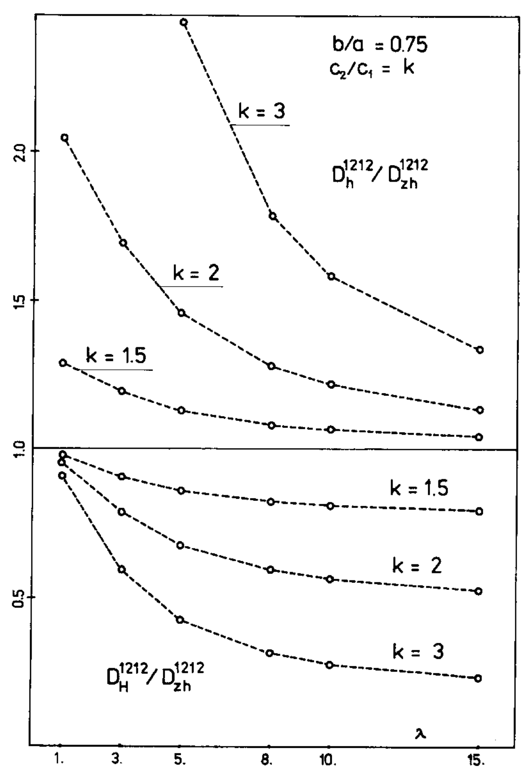
b/a = 0.75
c_2/c_1 = k
k = 3
D_h^{1212}/D_{zh}^{1212}
k = 2
k = 1.5
k = 1.5
k = 2
k = 3
D_H^{1212}/D_{zh}^{1212}
λ
2.0
1.5
1.0
0.5
1.
3.
5.
8.
10.
15.

Fig. 3.

when cells $\mathscr{Y}$ are longitudinally slender. On the other hand, the values D_H^{1212} underestimate the values D_{zh}^{1212} and do not tend to them as λ increases.

(c) Huber's formulae considerably overestimate the stiffness D^{1122}.

(d) Kączkowski's formulae overestimate the stiffness D^{2222}.

8. FINAL REMARKS

Of all the equations derived and discussed in the present paper, the formulae for the effective stiffnesses, which enter the primal homogenized (P_{hom}) problem, may merit special attention since they are of vivid interest for engineering practice. Although their asymptotic derivation can be viewed as formal, the present author shares the view of Kohn and Vogelius (1984) and Kalamkarov *et al.* (1987), that the formulae derived in Section of Part I (and discussed after) are correct. We feel confident that there is now no use in performing the smearing-out process in another manner. In the Caillerie-type approach discussed in the present paper, the effective model of a periodic plate is rationally and uniquely defined and, just contrary to the outdated opinion of Boot and Moore (1988), is not "an expedient which inevitably raises questions of validity". Thus, let it be stressed here that it is not a question of how to define the effective stiffnesses, but how to simplify the computations. In our opinion, the two-dimensional approximation proposed in Section 5 provides such a simplification. However, this approximation can only be applied to periodic plates whose periodicity cells also have the shapes of plates. When these periodicity cells are very thin plates, one can approximate the overall stiffnesses by using the hitherto existing formulae of Duvaut (1976). In the general case, one should compute the stiffness following the Galerkin-type algorithm described in Section 6 of Part II. Current computer programmes make it possible to implement this algorithm, thus enabling the variations of material and geometrical properties to be correctly taken into account.

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